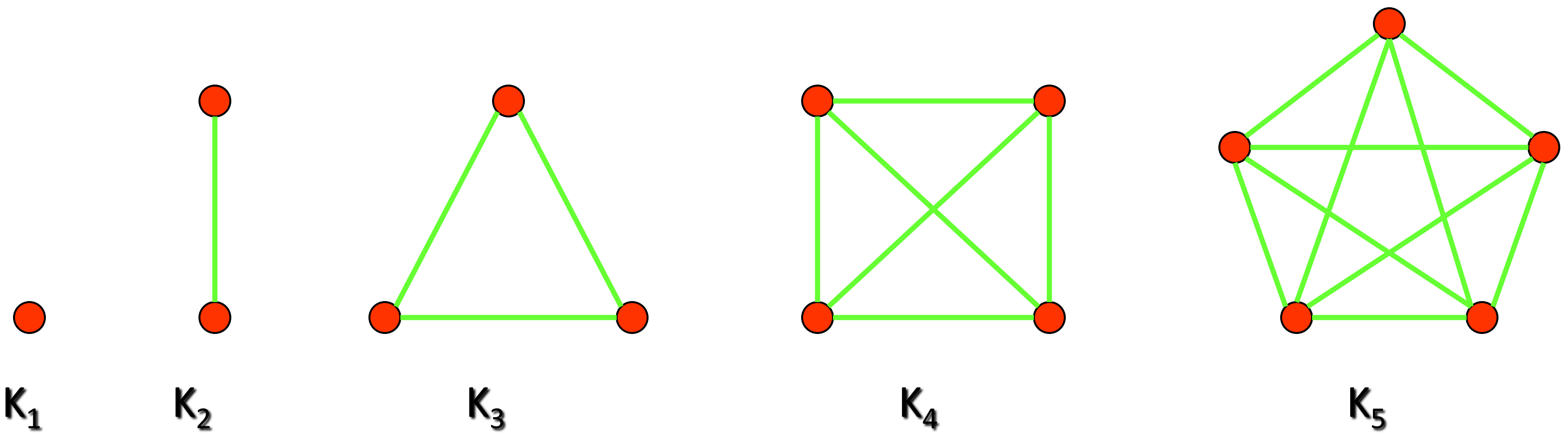
Graph Terminology

- **Theorem:** Let G = (V, E) be a graph with directed edges. Then:
- $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

- This is easy to see, because every new edge increases both the sum of in-degrees and the sum of out-degrees by one.

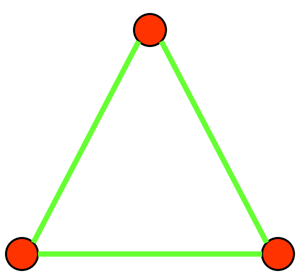
Special Graphs

- **Definition:** The **complete graph** on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

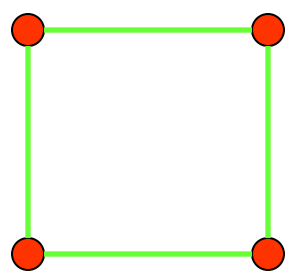


Special Graphs

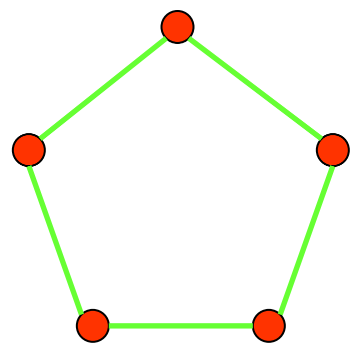
- **Definition:** The **cycle** C_n, $n \geq 3$, consists of n vertices v_1, v_2, …, v_n and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.



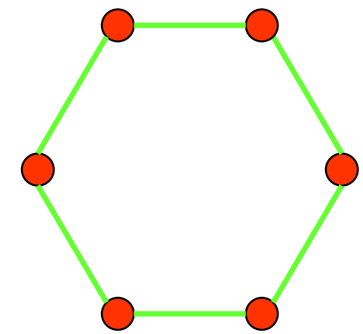
C_3



C_4



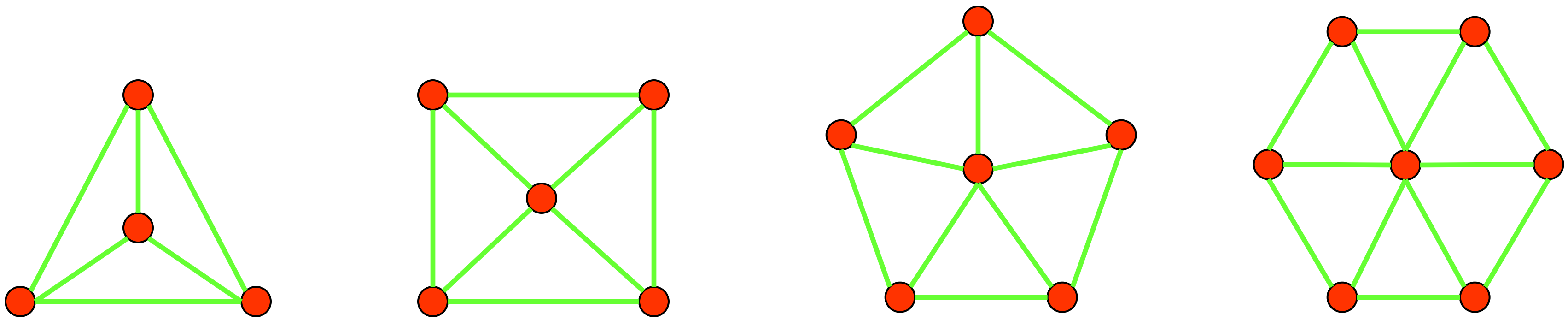
C_5



C_6

Special Graphs

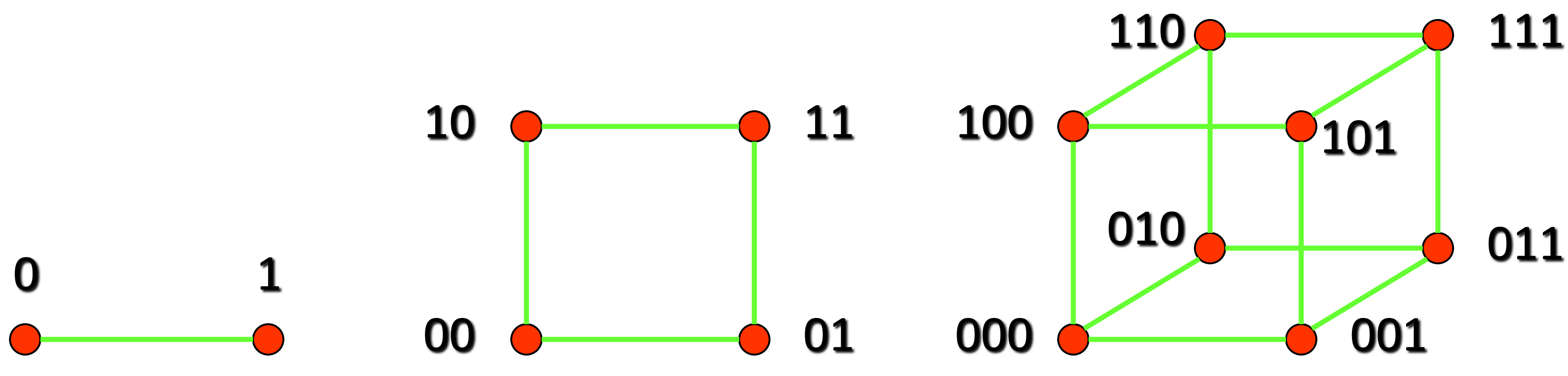
- **Definition:** We obtain the **wheel** W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n by adding new edges.



W_3 W_4 W_5 W_6

Special Graphs

•**Definition:** The **n-cube,** denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



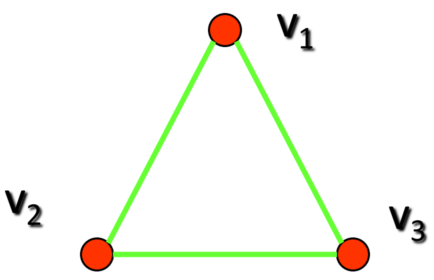
Q_1 Q_2 Q_3

Special Graphs

- **Definition:** A simple graph is called **bipartite** if its vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 with a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).
- For example, consider a graph that represents each person in a village by a vertex and each marriage by an edge.
- This graph is **bipartite**, because each edge connects a vertex in the **subset of males** with a vertex in the **subset of females** (if we think of traditional marriages).

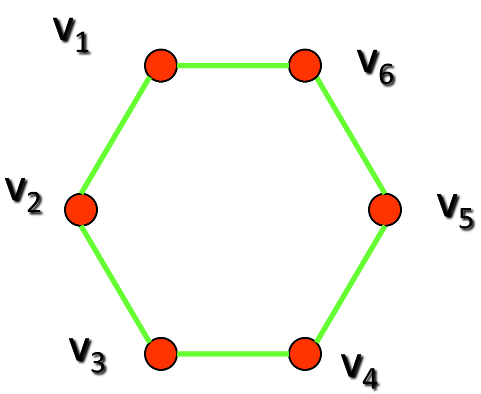
Special Graphs

- **Example I:** Is C_3 bipartite?

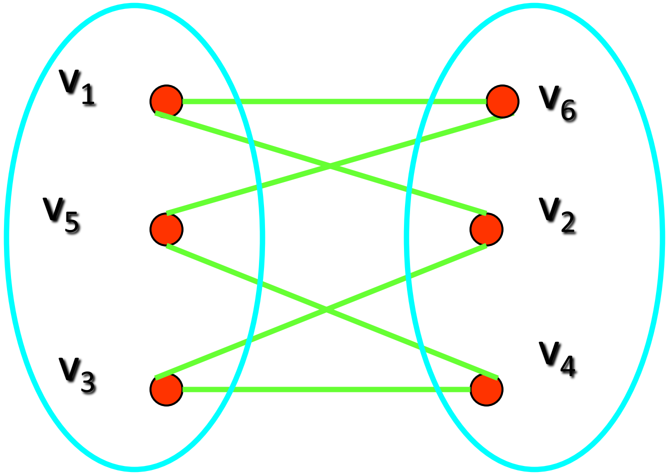


No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

Example II: Is C_6 bipartite?

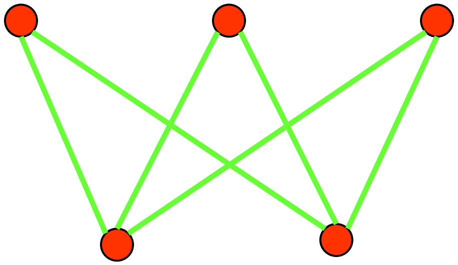


Yes, because we can display C_6 like this:

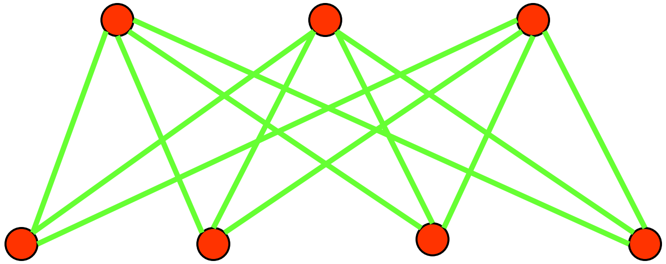


Special Graphs

- **Definition:** The **complete bipartite graph** $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. Two vertices are connected if and only if they are in different subsets.



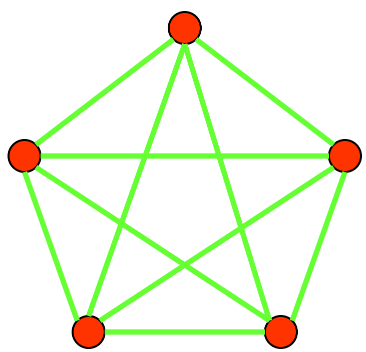
$K_{3,2}$



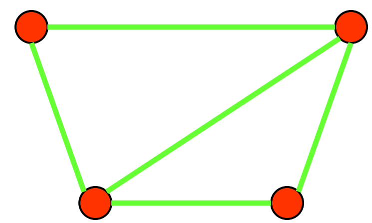
$K_{3,4}$

Operations on Graphs

- **Definition:** A **subgraph** of a graph G = (V, E) is a graph H = (W, F) where $W \subseteq V$ and $F \subseteq E$.
- **Note:** Of course, H is a valid graph, so we cannot remove any endpoints of remaining edges when creating H.
- **Example:**



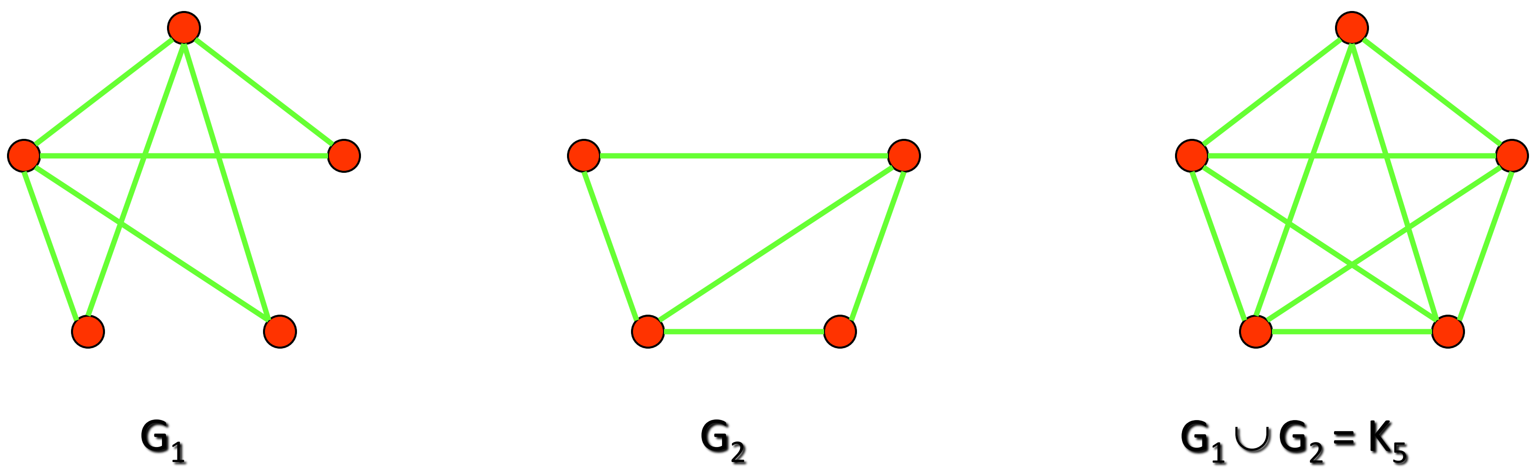
K_5



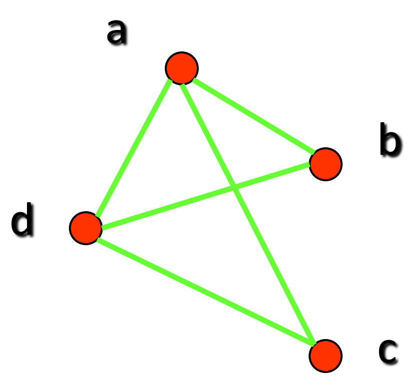
subgraph of K_5

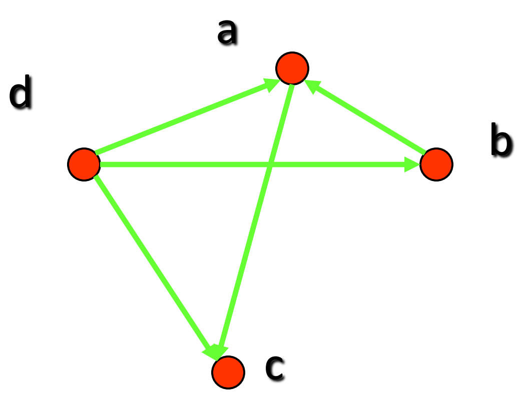
Operations on Graphs

- **Definition:** The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.
- The union of G_1 and G_2 is denoted by $\mathbf{G_1 \cup G_2}$.



Representing Graphs





Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Initial Vertex	Terminal Vertices
a	c
b	a
c	
d	a, b, c

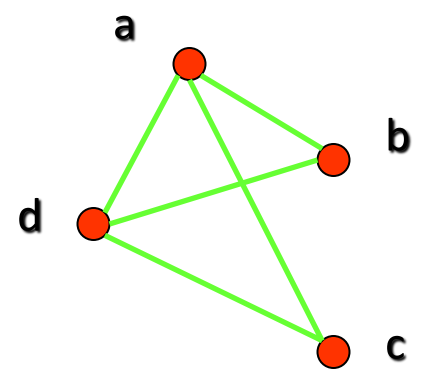
Representing Graphs

- **Definition:** Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.
- The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 otherwise.
- In other words, for an adjacency matrix $A = [a_{ij}]$,
- $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,

$a_{ij} = 0$ otherwise.

Representing Graphs

- **Example:** What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

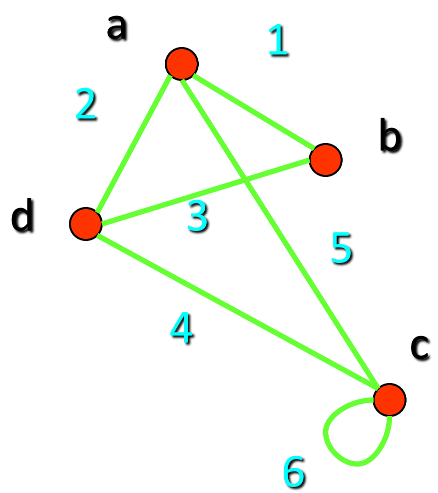
Note: Adjacency matrices of undirected graphs are always symmetric.

Representing Graphs

- **Definition:** Let $G = (V, E)$ be an undirected graph with $|V| = n$. Suppose that the vertices and edges of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$, respectively.
- The **incidence matrix** of G with respect to this listing of the vertices and edges is the $n \times m$ zero-one matrix with 1 as its (i, j)th entry when edge e_j is incident with v_i, and 0 otherwise.
- In other words, for an incidence matrix $M = [m_{ij}]$,
- $m_{ij} = 1$ if edge e_j is incident with v_i
  $m_{ij} = 0$ otherwise.

Representing Graphs

- **Example:** What is the incidence matrix M for the following graph G based on the order of vertices a, b, c, d and edges 1, 2, 3, 4, 5, 6?



Solution:

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Note: Incidence matrices of directed graphs contain two 1s per column for edges connecting two vertices and one 1 per column for loops.